Indian Statistical Institute, Bangalore
B. Math.(Hons.) I Year, Second Semester
Semestral Examination
Analysis -II (Back Paper)

Time: 3 hours Instructor: Pl.Muthuramalingam

Maximum Marks 50

1. a) For a bounded function $f : [a, b] \longrightarrow R$, define the upper sum $U(\mathbb{P}, f)$ and the lower sum $L(\mathbb{P}, f)$ for any partition $\mathbb{P}$. [1]

   b) If $\mathbb{P}_1 \supset \mathbb{P}$ and $\mathbb{P}_1$ has only one more point than $\mathbb{P}$ find some inequality between, $U(\mathbb{P}_1, f)$ and $U(\mathbb{P}, f)$ and prove it. [2]

   c) When is f Riemann integrable? [1]

   d) Show that f is Riemann integrable iff given $\varepsilon > 0$ there exists a partition $\mathbb{P}$ such that

   $$\mid U(\mathbb{P}, f) - L(\mathbb{P}, f) \mid \leq \varepsilon.$$

   [3]

   e) If f is a continuous function, show that f is Riemann integrable. [2]

2. Let $g : R^n \longrightarrow R$ be given by $g(x_1, x_2, \cdots, x_n) = x_1^2 + x_x^2 + \cdots + x_n^2$. For each $\vec{y}$ in R^n find the linear map $L_{\vec{y}} : R^n \longrightarrow R$ such that

   $$\frac{\mid g(\vec{y} + \vec{h}) - g(\vec{y}) - L_{\vec{y}}(\vec{h}) \mid}{\parallel \vec{h} \parallel} \longrightarrow 0$$

   as $\|\vec{h}\| \longrightarrow 0$ and prove your claim. [3]

3. State the chain rule for differentiation for functions of several variables. [2]

4. a) Define a metric space (X, d). [2]

   b) Define an open subset of (X, d). [1]

   c) Let G be open in (X, d) and $y \epsilon G$. If $x_n \longrightarrow y$ in (X, d), show that there exists n_0 such that $x_n \in G$ for all $n \geq n_0$. [2]

5. Show that every interval is a connected subset of R. [5]

6. Let $f : (X, d) \longrightarrow (Y, m)$ be a continues function between the metric spaces $(X, d), (Y, m)$. If (X, d) is compact show that f is uniformly continuous. [4]

7. If A_1, A_2 are compact subsets of a metric space show that $A_1 \cup A_2$ is also a compact subset. [2]

8. Let $f(x,y) = xy\,\frac{x^2-y^2}{x^2+y^2}$ on $R^2\{(0,0)\}$, $\qquad f(0,0) = 0$.

   Show that $\frac{\partial^2 f}{\partial x\, \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y\, \partial x}(0,0)$. [3]

9. Let $f : R^n \longrightarrow R$ be a function have all the derivatives of all orders and each of them is continuous. State Taylors expansion with a remainder term. [3]

10. Let $g : R^2 \longrightarrow R$ be given by $g(x,y) = x^2 - y^2 + 100$. Calculate $g, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ at $(0,0)$. Show that there exist sequences $a_n \longrightarrow 0, b_n \longrightarrow 0$ with $g(a_n) > g(0) > g(b_n)$. [2]

11. Let (X,d) be a metric space with $d(x,y) = 0$ for $x = y, d(x,y) = 1$ for $x \neq y$. Show that every subset is an open subset. [1]

12. a) For any matrix $A = ((a_{ij}))i = 1,2,\cdots,n, j = 1,2,\cdots,k,\ a_{ij}$ real define $\parallel A \parallel$ by $\parallel A \parallel = [\sum\limits_{i,j} \mid a_{ij} \mid^2]^{\frac{1}{2}}$. If AB are matrices such that A, B is also a matrix show that

$$\parallel AB \parallel \leq \parallel A \parallel \parallel B \parallel .$$

[2]

b) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $\|AB\| \neq \parallel A \parallel$ $\parallel B \parallel$. [1]

c)$A_k, A, B_k, B \epsilon M_{n\times n}(R)-$ the space of $n \times n$ real matrices. If $\parallel A_k - A \parallel$ $+ \parallel B_k - B \parallel \longrightarrow 0$ as $k \longrightarrow \infty$ then show that $\parallel A_k B_k - AB \parallel \longrightarrow 0$ as $k \longrightarrow \infty$. [2]

d) Let $G_1, G_2 : M_{n\times n}(R) \longrightarrow M_{n\times n}(R)$ have total derivative at X_0. Define $F : M_{n\times n}(R) \longrightarrow M_{n\times n}(R)$ by $F(X) = G_1(X)G_2(X)$.Let the error functions $E_1(X_0,U), E_2(X_0,U), E(X_0,U)$ for U in $M_{n\times n}(R)$ be given by

$$E_1(X_0,U) = G_1(X_0+U) - G_1(X_0) - G_1'(X_0)U,$$
$$E_2(X_0,U) = G_2(X_0+U) - G_2(X_0) - G_2'(X_0)U,$$

$E(X_0,U) = F(X_0+U) - F(X_0) - G_1'(X_0)UG_2(X_0) - G_1(X_0)G_2'(X_0)U.$

Verify that $E(X_0,U) =$

$E_1(X_0,U)G_2(X_0+U) + G_1(X_0)E_2(X_0,U) + G_1^1(X_0)U[G_2(X_0+U) - G_2(X_0)]$

or verify that $E(X_0, U) =$

$$G_1(X_0+U)E_2(X_0, U)+E_1(X_0, U)G_2(X_0)+[G_1(X_0+U)-G_1(X_0)]G_2^1(X_0)U.$$

[3]

e) Show that F has a total derivative at X_0. Find $F'(X_0)U$ in terms of $X_0, U, G_1, G_2, G_1', G_2'$. [3]